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OPTIMIZATION OF SEEPAGE RATE THROUGH A TRIANGULAR CORE

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SUMMARY

Seepage through a triangular dam core is studied by the hodograph method. Core slope providing minimal seepage rate at prescribed head value and core cross-sectional area is found. A simple flow pattern involving seepage face, constant head, and non-flow boundaries is assumed. Seepage through a cake of low permeable sediments deposited along the bottom of a channel is treated analogously. © 1997 by John Wiley & Sons, Ltd.

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1. INTRODUCTION

Low permeable cores are used to reduce seepage through earth dams.[1,2] Selection of core construction features, i.e. its geometrical sizes and material, is an important step in dam design which includes estimations of efficiency against seepage measured usually in terms of water losses and stability characterized by hydraulic gradients in zones of potential failure. Numerical methods like FDM and FEM provide powerful procedures to treat seepage in heterogeneous porous massifs involving phreatic surfaces, unsaturated zones, deviations from Darcy's law and other complications. However, for preliminary estimations simple analytical formulae are still of some interest. Even though these formulae are based on idealized flow-matrix assumptions they provide test examples for more realistic models and, and it is well-known, serve for training and teaching, developing 'short-cut' methods, fast account of main trends, 'back-of-an envelope' calculations, and on-site applications with low computer capacity.

In this note, seepage through a triangular core of an earth dam is considered analytically. Flow through a low permeable sediment layer at the bottom of a channel is treated analogously. The following questions are raised: Does a core of best counter-seepage property under reasonable restrictions exist? How fast do hydraulic gradients increase near the points of potential erosion? To answer these questions we solve two isoperimetric problems. Namely, at prescribed core cross-sectional area and head in the dam upper pool the slope angle providing minimal seepage

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rate through the core is found. At a prescribed channel area, seepage losses are minimized by slope angle variations. Hydraulic gradients along the seepage face are calculated. Seepage is assumed to be Darcian, steady, 2D and fully saturated.

In the classical monographs[3–6] only specific cases of the problem are presented. We use the standard conformal mapping of a triangle in the physical plane onto a finite or infinite triangle in the hodograph plane and develop an explicit solution for all slope angles.

2. SEEPAGE THROUGH A CORE

Consider an earth dam with a low permeable core ABC placed on an impermeable horizontal bottom. Generally, seepage flow involves phreatic surfaces within the dam core, unsaturated zones and stream line refraction at the interfaces between areas of different conductivity (Figure 1). It seems impossible to treat analytically the whole flow and the problem is usually decomposed into a number of elementary blocks. For example, the capillary syphoning in unsaturated zone II (Figure 1) was described qualitatively[7,8] and considered in terms of Vedernikov and Richards models under simplified conditions.[9] Zone IV can be treated as the usual saturated phreatic surface flow with distributed accretion.[4] Zone III is a domain of 'infiltration rain'.[6] Zone V where flow exhibits 2-D transition between areas of distinct conductivity was studied thoroughly in electrodynamics[10] as a 'refraction' wedge. In what follows, we study one element of the complex picture, seepage within the dam core I. We assume that AB is a constant head boundary. This approximation holds if core conductivity k is much lower than dam conductivity k_d. We ignore zone IV, i.e. assume that water which seeped through the core is removed effectively. At last we assume the whole core is fully saturated, that is true if: (a) water level in the upper pool equals the core height H, (b) angle $\gamma < \pi/2$ (Reference 6). Note that BC can be modelled as a seepage face if a chimney drain is installed along this boundary or $k \ll k_d$ (see Cedergen,[1] Figure 6.12(d)) and Figure 6.17(a)). Thus, we come to the flow pattern shown in Figure 2 which was investigated by Nelson–Skornjakov[5] and referenced by Polubarinova-Kochidna.[6] Unfortunately, Nelson–Skornjakov's approximate formulae do not give correct results and we had to re-derive the analytical solution.

The main goal of the core is reduction of seepage which is characterized by the total seepage rate, Q. A characteristic of core material amount is the area, S. We fix S, the head value H (hence, the dam width L) and solve the following problem:

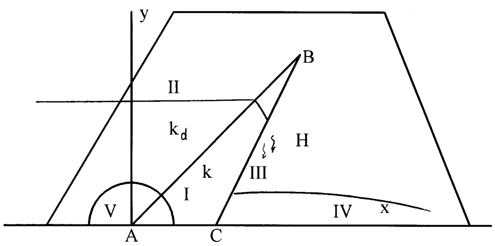


Figure 1. Seepage zones in an earth dam

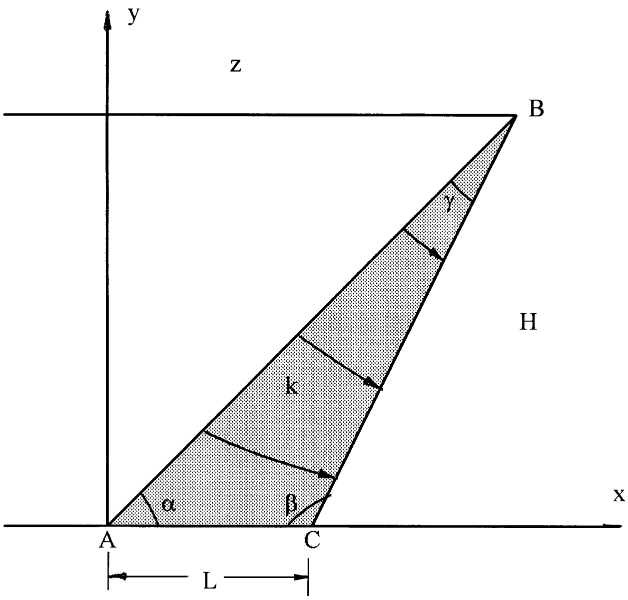


Figure 2. Flow through a triangular core

Problem 1. Define the slope angle α providing minimal seepage rate Q at prescribed H, S, k

Solution of this problem if unique and global can serve for isoperimetric estimations and comparisons.[11,12]

For $\alpha < \pi/2$ and $\beta < \pi/2$, the hodograph domain is a finite triangle (Figure 3(a)). For $\alpha < \pi/2$ and $\beta > \pi/2$, velocity at the point C is infinite (Figure 3(b)), while for $\alpha > \pi/2$ and $\beta < \pi/2$, the point A exhibits infinite velocity (Figure 3(c)). At $\alpha = \pi/2$, the flow is 1-D and hodograph degenerates into one point.

Conformal mapping of the triangle in the physical plane z, onto the triangle in the hodograph plane, V (Figure 3) yields the following formula for Q:

$$\frac{Q}{kH} = \frac{U_B}{\sin^2\alpha\, B(g,a)} \int_0^1 x^{g-1}(1-x)^{a-1}\left[1 - \frac{B_x(a+b, 0{\cdot}5-a)}{B(a+b, 0{\cdot}5-a)}\right] \mathrm{d}x, \quad \text{at } \alpha < \pi/2 \tag{1}$$

$$\frac{Q}{kH} = \frac{U_B}{\sin^2\alpha} - \frac{U_B}{\sin^2\alpha\, B(g,a)\, B(a+b, 0{\cdot}5-a)}$$

$$\times \int_0^1 x^{g-1}(1-x)^{a-1} B_x(a+b, 0{\cdot}5-a)\,\mathrm{d}x, \quad \text{at } \alpha > \pi/2$$

$$U_B = \frac{1}{\cot\alpha + \cot\beta} = \frac{H}{L}$$

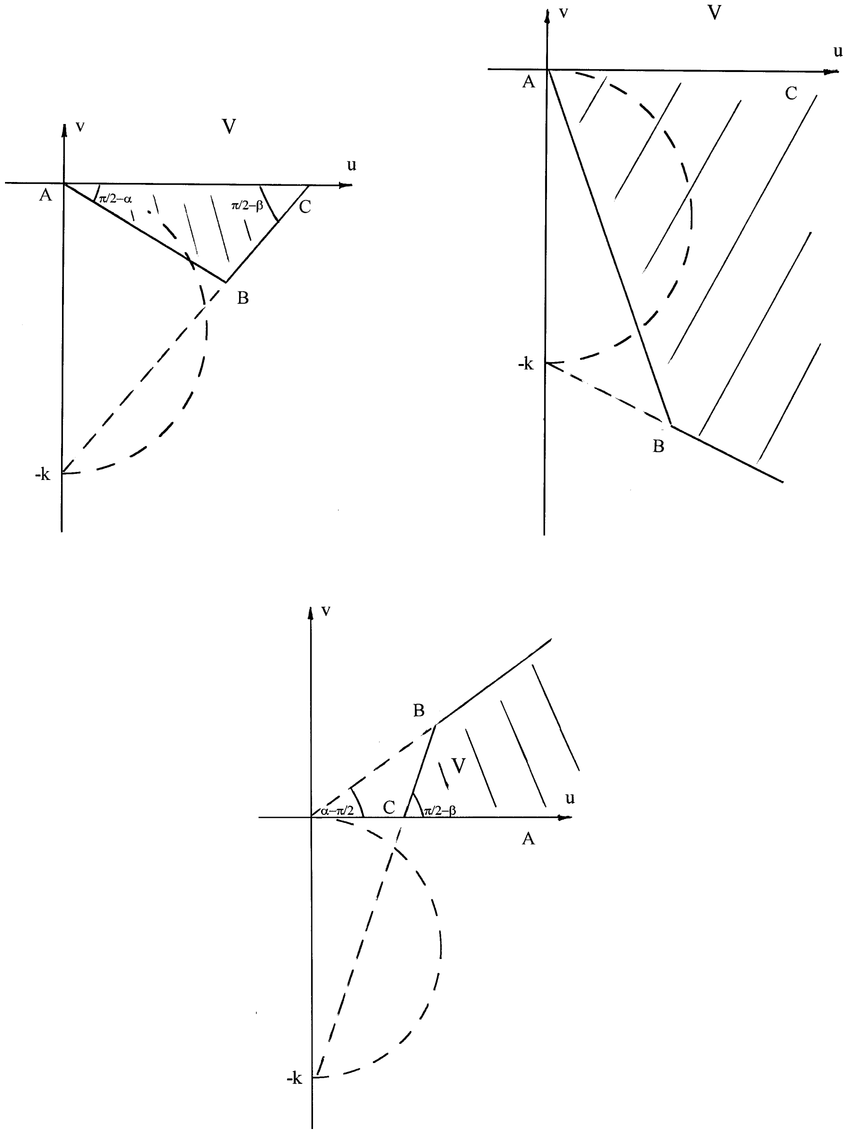


Figure 3. Hodograph domains for (a) $\alpha < \pi/2$, $\beta < \pi/2$, (b) $\alpha < \pi/2$, $\beta > \pi/2$, (c) $\alpha > \pi/2$

where $B(a, b)$ and $B_x(a, b)$ designate the complete and incomplete *beta*-functions,[13] $a = \alpha/\pi$, $b = \beta/\pi$, $\gamma = \pi - \alpha - \beta$, $g = \gamma/\pi$. Note that $\beta = \cot^{-1}[L/H - \cot(\alpha)]$, $0 < \beta < \pi$ and Problem 1 is reduced to minimization of $Q(\alpha)$.

Figure 4 shows the graphs of $Q^* = Q/kH$ as functions of a plotted for $L/H = 1{\cdot}0, 1{\cdot}5, 2{\cdot}0$ (curves 1–3, respectively). Solution of the Problem 1 is derived by the unique minima $Q^*_{\min}$ and $\alpha_{\min}$ in the graphs. For the data in Figure 4, $Q^*_{\min} \approx 0{\cdot}957, 0{\cdot}616, 0{\cdot}446$ and $\alpha_{\min} \approx 76, 70, 65°$.

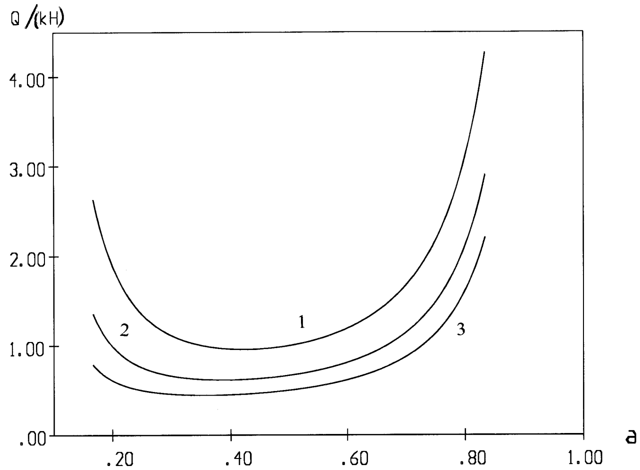


Figure 4. Seepage rate Q/kH as function of angle α/π for $L/H = 2{\cdot}0$, $1{\cdot}5$, $1{\cdot}0$ (curves 1–3, respectively)

Two explicit formulae for special slope angles fit our results. Namely, at $\alpha = \beta = \pi/4$ the rate $Q^* = 0{\cdot}5$ (Reference 4, pp. 393–394). At $\alpha = \pi/2$ all stream lines are horizontal and $Q^* = \tan\beta$ (Reference 6, p. 286).

The value of hydraulic gradient $G = |V|/k = \sqrt{u^2 + v^2}/k$ along BC is used to estimate slope stability.[1] Obviously, at $\alpha < \pi/2$ with imposed restriction $\gamma < \pi/2$ the value of G increases monotonously from point B to C while at $\alpha > \pi/2$ the opposite is true. For $\alpha < \pi/2$ the horizontal and vertical components of velocity as functions of y can be written in the following parametric form ($0 \leqslant t \leqslant 1$):

$$\frac{u}{k} = U_B + \frac{U_B \sin\beta\,[B(0{\cdot}5 - b, a + b) - B_t(0{\cdot}5 - b, a + b)]}{\sin\alpha\, B(a + b, 0{\cdot}5 - a)} \tag{2}$$

$$v = -k + u\cot\beta$$

$$\frac{y}{H} = 1 - \frac{\sin\beta\,[B(b, a + b) - B_t(b, a + b)]}{\sin\alpha\, B(g, a)}$$

For $\alpha > \pi/2$ velocity distributions are similar to equation (2).

Figure 5 illustrates functions $G(y/H)$ for $L/H = 2.0$, $\alpha = 20, 40, 60^\circ$ (curves 1–3, respectively) with infinite gradient at the point C. Similar distributions were established from conformal mappings of triangles (half-strips) in the physical and complex plane.[14]

3. SEEPAGE THROUGH A CHANNEL CAKE

The solution derived can be easily transformed to the case of seepage from a channel B_1AB which bottom is covered by a low permeable sediment cake (Figure 6(a)). It is well-known that seepage from streams and channels is often controlled by the sediment layer whose conductivity k is much lower than that of the ambient ground k_g.

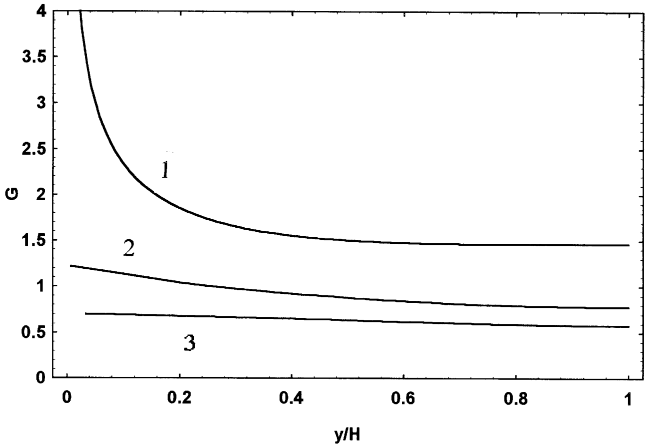


Figure 5. Hydraulic gradient distributions along BC for $L/H = 2{\cdot}0$, and $\alpha = 20, 40, 60^{\circ}$ (curves 1–3, respectively)

Consider the right part of a triangular channel with maximal water depth H, width $2L$, slope α, and area $S = LH$. Assume that the cake thickness d is proportional to the channel depth h, i.e. $d = \varepsilon h$, $\varepsilon =$ const that approximates the intuitively clear dependence of deposition rate on water height. Thus, BC is a straight line with slope angle $\beta = \tan^{-1}[(1+\varepsilon)\tan\alpha]$. Specify along BC the usual seepage face condition.[3] Then in the physical plane the flow domain is a triangle and in the hodograph plane the corresponding domain is an infinite triangle (Figure 6(b)). Further analysis is analogous to the one presented above. In particular, we solve the following problem.

Problem 2. Define the slope angle α providing minimal seepage rate Q at prescribed S, k, ε

A similar problem was solved for phreatic surface flow from a channel without cake[15,16] by minimization of the shape factor $\mu = Q/(k\sqrt{S})$. In the case under study:

$$\frac{Q}{k} = U_B(H + L\cot\alpha) + \frac{U_B(H\cos\gamma + L\sin\gamma) + (V_B+1)(H\sin\gamma - L\cos\gamma)}{B(\beta, 1-\gamma)\,B(\gamma, 0{\cdot}5-\alpha)}E \tag{3}$$

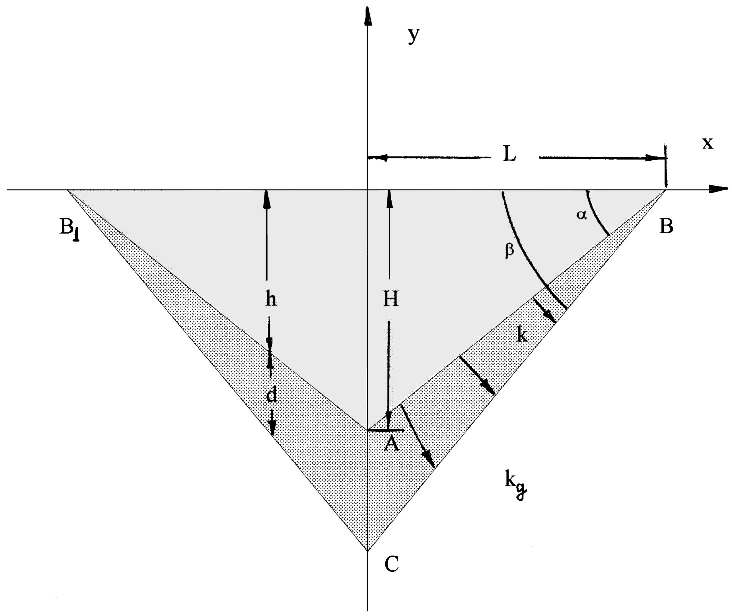
$$E = \int_0^1 x^{g-1}(1-x)^{a-0{\cdot}5}B_x(1-g, -a)\,\mathrm{d}x$$

$$U_B = \frac{1}{\cot\alpha - \cot\beta}, \quad V_B = -U_B\cot\alpha$$

where $a = \alpha/\pi$, $b = \beta/\pi$, $\gamma = \beta - \alpha, g = \gamma/\pi$.

Figure 7 shows the function $\mu(a)$ for $\varepsilon = 0{\cdot}1$ which minimum $\mu_{\min} \approx 39$, $\alpha_{\min} \approx 28^{\circ}$ defines solution of Problem 2.

Note that the condition of atmospheric pressure along BC in the cake problem is a serious approximation since, in contrast with the dam problem, artificial sinks along BC (like



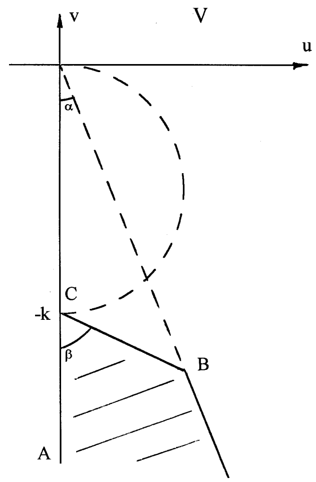


Figure 6. Seepage from a channel with a sediment layer: (a) physical plane, (b) hodograph plane

horizontal drains) are rarely installed (for example, under landfill liners when percolated fluid can contaminate ground water while chimney or other drains in dams are of vital importance because of stability criteria). It means that below BC, seeped water has to infiltrate downward without restrictions. A necessary (not sufficient!) condition of this infiltration is $(kU_B)^2 + (kV_B)^2 < k_g^2$.

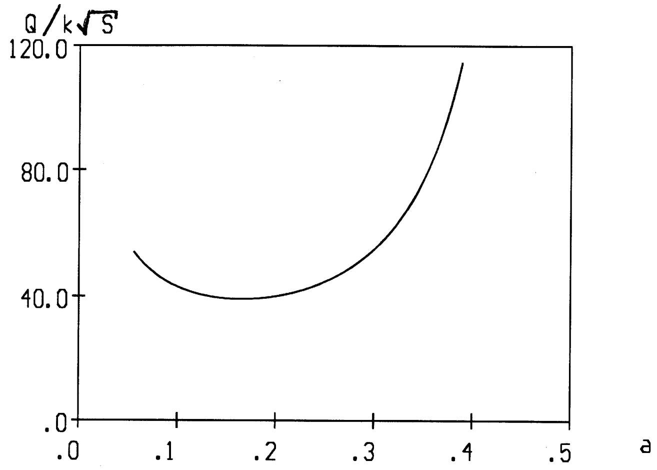


Figure 7. Non-dimensional seepage losses from a channel as function of slope angle

4. DISCUSSION

Equations (1)–(3) and corresponding graphs can serve for estimations of seepage at more complex boundary conditions. For example, if a part of the core segment BC is a constant head line (i.e. zone IV is presented) then Q will be lower than equation (1) predicts. The same is true if a free surface appears in the core, i.e. water level in the upper pool drops below the core tip or $\gamma > \pi/2$. These and other estimations, both for Q values and other seepage characteristics (G, hydraulic head, pressure), can be done on the basis of the general comparison theorems.[17,12] Simple stability criteria can be tested on the basis of graphs in Figure 5 like $G < J_{cr}$ along BC where J_{cr} is a critical value.[2]

Recall the restriction $\gamma < \pi/2$ (or $L/H < 2$) that guarantees full saturation of the core. Nevertheless, the hodograph method works also if a phreatic surface appears and hodograph domain becomes a pentagon.[6]

Flow patterns studied along with other simple analytical solutions for core or liner zones[18–20] can provide input values (velocity distributions along BC) for analytical models describing seepage in unsaturated zones.

NOTATION

H	acting head
k	hydraulic conductivity
L	core width
Q	total flow rate
S	cross-sectional area
$V = u + iv$	seepage velocity
$z = x + iy$	physical coordinate
α, β, γ	core angles

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